Ekeland's Theorem

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1 Notations

lets denote :

- (E,d), any complete metric space.
- H, any Hilbert space.
- B, any Banach space.
- X, any smooth Banach space : X is a Banach space where there exist $\phi : X \mapsto \mathbb{R}^+$ such that $\Delta = \{x, \phi(x) > 0\}$ is a bounded non empty set on which ϕ is Fréchet differentiable.
- if $f : E \mapsto \mathbb{R}$, then $Epi(f) = \{(x,t) \in E \times \mathbb{R}, t \geq f(x)\}$
- $\mathcal{C}((x_0,t_0),\alpha)$, the lower cone of vertex (x_0,t_0) and parameter α in $E \times \mathbb{R}$:

$$\mathcal{C}((x_0,t_0),\alpha) = \{(x,t) \in E \times \mathbb{R}, |t - t_0| \leq \alpha d(x,x_0)\}$$

2 The Theorem

Theorem 2.1 *Let f be a lower semicontinuous function, with a lower bound L, from E onto* $\mathbb{R}$*. Then* $\forall x \in E, \forall \varepsilon \in \mathbb{R}, \exists \overline{x} \in E$ *with the two following properties :*

- $d(x,\overline{x}) \leq \frac{\lambda}{\varepsilon}$
- $\mathcal{C}((\overline{x}, f(\overline{x}), \varepsilon) \cap Epi(f) = \{(\overline{x}, f(\overline{x}))\}$

where $\lambda = f(x) - Inf(f)$

It is a basic application of the decreasing compact sequence theorem in complete metric spaces (if $\alpha > 0$, then $\mathcal{C}(y, f(y)) \cap Epi(f)$ is a compact non empty set $\forall y \in E$).

3 First results

- a demonstration of Picard-Banach fixed point theorem
- Caristi's theorem (1976)

4 Fréchet Differentiability of convex functions of Hilbert spaces

Theorem 4.1 *If f is a continuous convex function from H onto $\mathbb{R}$, then it is Fréchet differentiable anywhere on a generic set G.*

let :

- $\mathcal{M} = \{(u,a) \in H \times \mathbb{R}, f(x) \geq a - (u|x), \forall x \in H\}$
- $\mathcal{U}(x,c) = \{u \in H, \exists a \in \mathbb{R} : (u,a) \in \mathcal{M}$ and $a - (u|x) \geq f(x) - c\}$
- $P_\varepsilon = \{x \in H, \exists v \in H, \exists \eta > 0 : \|y - x\| \leq \eta \Rightarrow f(y) \leq f(x) - (v|y-x) + \varepsilon \|y-x\|\}$
- $\Lambda_c = \{(u_i, m_i) \in \mathcal{M}$ and $m_1 - (x_i|x) \leq f(x) - c\}$
- $G_\varepsilon = \{x \in H : \exists c > 0, diam(proj_H(\Lambda_c)) \leq \varepsilon\}$

Then the proof works with following steps :

- $\forall x \in H, c > 0$, $\mathcal{U}(x,c)$ is bounded
- $\forall \varepsilon > 0$, P_ε is dense into H
- if $x \in \bigcap_{\varepsilon>0} G_\varepsilon$, then f is Fréchet differentiable at x.
- $\forall \varepsilon > 0$, G_ε is an open set
- $\forall \varepsilon > 0$, $\exists \varepsilon' > 0$, $P_{\varepsilon'} \subset G_\varepsilon$
- by Baire's dense open set theorem, this theorem follows.

Remark 4.2 *This theorem can be generalized in smooth Banach spaces. For example if B is reflexive or B^* separable, then it is smooth (l^1,l^∞).*